

Convergence of Siegel-Veech constants

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Abstract

We show that for any weakly convergent sequence of ergodic $SL_2(\mathbb{R})$ -invariant probability measures on a stratum of unit-area translation surfaces, the corresponding Siegel-Veech constants converge to the Siegel-Veech constant of the limit measure. Together with a measure equidistribution result due to Eskin-Mirzakhani-Mohammadi, this yields the (previously conjectured) convergence of sequences of Siegel-Veech constants associated to Teichmüller curves in genus two.

The proof uses a recurrence result closely related to techniques developed by Eskin-Masur. We also use this recurrence result to get an asymptotic quadratic upper bound, with a uniform constant depending only on the stratum, for the number of saddle connections of length at most R on a unit-area translation surface.

1 Introduction

1.1 Setting

Basic Definitions. A *translation surface* is a pair $X = (M, \omega)$, where M is a Riemann surface, and ω is a holomorphic 1-form. Away from its zeroes, ω defines a flat (Euclidean) metric. The metric has a conical singularity of cone angle $2(n+1)\pi$ at each zero of order n .

A *saddle connection* is a geodesic segment that starts and ends at zeroes (we allow the endpoints to coincide), with no zeroes on the interior of the segment. We can also consider closed loops not hitting zeroes that are geodesic with respect to the flat metric. Whenever there is one of these, there will always be a continuous family of parallel closed geodesic loops with the same length. We refer to a maximal such family as a *cylinder*. Every cylinder is bounded by a union of saddle connections parallel to the cylinder.

The bundle $\Omega\mathcal{M}_g$ of holomorphic 1-forms over \mathcal{M}_g (the moduli space of genus g Riemann surfaces), with zero section removed, can be thought of as the moduli space of translation surfaces. This bundle breaks up into strata of translation surfaces that have the same multiplicities of the zeroes of ω . We denote by $\mathcal{H}(m_1, \dots, m_k)$ the stratum of surfaces with k zeroes of order m_1, \dots, m_k , and we let $\mathcal{H}_1(m_1, \dots, m_k)$ be the subset of all the unit-area surfaces (area is measured with respect to the flat metric).

Action of $SL_2(\mathbb{R})$. By cutting along saddle connections, we can represent every translation surface as a set of polygons in the plane, such that every side is paired up with a parallel side of equal length. Some of the vertices may have more than 2π angle around them after all the gluings of paired sides are made - these are singularities of the metric.

Since $SL_2(\mathbb{R})$ acts on polygons in the plane, preserving the property of a pair of sides being parallel and equal length, the group acts on the space of translation surfaces. In fact, it acts on each stratum \mathcal{H} , and even on a unit-area stratum \mathcal{H}_1 since the action preserves area. On each \mathcal{H}_1

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there is a canonical probability measure μ_{MV} in the Lebesgue measure class, called the *Masur-Veech measure*, which is $SL_2(\mathbb{R})$ -invariant and in fact ergodic ([Mas82], [Vee82]). This measure is defined in terms of the periods of the 1-form ω .

There is a rich interplay between dynamics on an individual translation surface X (e.g. properties of saddle connections or cylinders) and the dynamics of the $SL_2(\mathbb{R})$ action on strata. In particular, by the seminal work of Eskin-Mirzakhani [EM13] and Eskin-Mirzakhani-Mohammadi [EMM15], the orbit closure $\overline{SL_2(\mathbb{R})X}$ supports a canonical $SL_2(\mathbb{R})$ -invariant ergodic probability measure, and properties of this measure are closely connected to dynamics on individual translation surfaces.

Siegel-Veech constants. Let $N(X, R)$ be the number of cylinders on X of length at most R . The study of asymptotics of this function as $R \rightarrow \infty$ is of central importance and has inspired much of the work on spaces of translation surfaces. By work of Masur ([Mas88] and [Mas90]), for a fixed X , there are quadratic upper and lower bounds for the growth of $N(X, R)$ in terms of R . If μ is an ergodic $SL_2(\mathbb{R})$ -invariant probability measure on \mathcal{H}_1 , then by Eskin-Masur ([EM01]) there exists a constant $c(\mu)$, the *cylinder Siegel-Veech constant* associated to μ , characterized by the property that for μ -a.e. X in \mathcal{H}_1 ,

$$N(X, R) \sim c(\mu) \cdot R^2$$

as $R \rightarrow \infty$.

For more information on translation surfaces, the reader can consult one of many surveys available on the topic, for instance [Zor06] or [Wri15].

1.2 Convergence of Siegel-Veech constants

The first new result states that if measures converge, then the corresponding Siegel-Veech constants do as well.

Theorem 1.1. *Suppose μ_1, μ_2, \dots are ergodic $SL_2(\mathbb{R})$ -invariant probability measures on \mathcal{H}_1 , and that $\mu_n \rightarrow \eta$, in the weak-* topology, where η is another ergodic $SL_2(\mathbb{R})$ -invariant probability measure. Then the Siegel-Veech constants satisfy $c(\mu_n) \rightarrow c(\eta)$.*

In Section 2, we use Theorem 1.1 to prove convergence, for the stratum $\mathcal{H}_1(2)$, of Siegel-Veech constants for non-arithmetic Teichmüller curves (numerical evidence of this was found by Bainbridge [Bai07]), and for arithmetic Teichmüller curves (conjectured by Lelièvre, based on numerical evidence and proof in a restricted case [Lel06]).

Remark 1.1. We will work with the cylinder Siegel-Veech constant for concreteness, but the result and proof work for other Siegel-Veech constants as well, in particular for the saddle connection Siegel-Veech constant (which counts saddle connections rather than cylinders) and the area Siegel-Veech constant (which is formed from counts of cylinders weighted by the area of the cylinder). The result and proof also works with a pair (X, q) , where q is a holomorphic *quadratic* differential (also known as a *half-translation surface*).

Remark 1.2. Note that if we define the Siegel-Veech constant of X to be the Siegel-Veech constant of the canonical measure whose support is the orbit closure $\overline{SL_2(\mathbb{R})X}$, then this does *not* define a continuous function on \mathcal{H}_1 . This is because special surfaces with small orbit closure, for instance Veech surfaces, will often have Siegel-Veech constants different from $c(\mu_{MV})$, the Siegel-Veech constant for the Masur-Veech measure on \mathcal{H}_1 (see Section 1.5 for references), while a dense subset of surfaces will have Siegel-Veech constant equal to $c(\mu_{MV})$.

1.3 Uniform asymptotic quadratic upper bound

The second new result gives a *uniform* quadratic upper bound on the number of cylinders, which holds, asymptotically, for *all* surfaces in a stratum.

Theorem 1.2. *Given \mathcal{H}_1 a unit-area stratum, there exists a constant c_{\max} such that for any surface $X \in \mathcal{H}_1$,*

$$N(X, R) \leq c_{\max} R^2$$

for all $R \geq R_0(X)$, where $R_0 : \mathcal{H}_1 \rightarrow \mathbb{R}$ is an explicit function of the length of the shortest saddle connection on X (and of the genus of the stratum).

Note that the function R_0 will not in general be bounded for a fixed stratum. This is because a surface in a fixed stratum can have arbitrarily many short saddle connections (for instance, by taking a surface with a short slit, and then gluing in a cylinder with small height and circumference). But, according to Theorem 1.2, as we increase R , the initially higher count caused by these short saddle connections eventually diminishes.

Remark 1.3. As for Theorem 1.1, the result and proof works if we replace the count of cylinders with the count of saddle connections, or the count of cylinders weighted by the area of the cylinder. The result and proof also work with a pair (X, q) , where q is a holomorphic quadratic differential.

1.4 Recurrence result for the proofs

Let $\ell(X)$ denote the length of the shortest saddle connection on X , and let

$$g_t = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}, \quad r_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

The main tool needed in the proofs of both Theorem 1.1 and Theorem 1.2 is the following recurrence-type result, which controls the length of the shortest saddle connection, on average over translation surfaces on a large “circle” centered at any X .

Proposition 1.1. *For any stratum \mathcal{H}_1 and $0 < \delta < 1/2$, there exists a function $\alpha : \mathcal{H}_1 \rightarrow \mathbb{R}_{\geq 0}$ and constants c_0, b such that for any $X \in \mathcal{H}_1$,*

$$\int_0^{2\pi} \frac{1}{\ell(g_T r_\theta X)^{1+\delta}} d\theta \leq c_0 e^{-(1-2\delta)T} \alpha(X) + b,$$

for all $T > 0$. The function $\alpha(X)$ is bounded above by an explicit function of $\ell(X)$ (and the genus of the stratum).

Closely related results appear in [EM01] and [Ath06] (see also [EMM15], Theorem 4.1). When we use Proposition 1.1, it will be crucial that the constant b does not depend on the surface X .

1.5 Previous work

A good deal of progress has been made in understanding the Siegel-Veech constants of Veech surfaces, which often lead to explicit expressions for the quadratic growth rates for billiards on polygons. In his foundational paper [Vee89], Veech used Eisenstein series to show that all Veech surfaces satisfy an exact quadratic asymptotic for the growth rate of cylinders, and he gave a way of computing the constants. He computes the constants for translation surfaces arising from unfolding certain isosceles triangles. Gutkin-Judge [GJ00] give a different formula for computing the Siegel-Veech constant, the proof of which uses softer ergodic-theoretic results related to counting horocycles in the hyperbolic plane. Vorobets [Vor96] discovered similar results independently.

Schmoll [Sch02] studied the problem of counting cylinders and saddle connections on tori with additional marked points. Along similar lines, Eskin-Masur-Schmoll [EMS03] study translation surfaces that are branched covers of tori. Using Ratner theory, they get exact quadratic asymptotics for all these surfaces, and they explicitly compute the constants for certain surfaces arising from

billiards in rectangles with barriers. In complementary work, Eskin-Marklof-Morris [EMWM06] study the case of branched covers of Veech surfaces that are not tori. They also get exact quadratic asymptotics for these surfaces, and they explicitly compute the constants for certain (non-Veech) surfaces that arise from unfolding triangles. Their proof modifies the techniques of Ratner to work in their setting, where one doesn't quite have a homogeneous space.

The results discussed above apply only to special translation surfaces. In the opposite direction, one can ask about Siegel-Veech constants for the Masur-Veech measure on a whole-stratum. Eskin-Masur-Zorich [EMZ03] give a general method for computing these in terms of the volumes of strata and neighborhoods of certain parts of the boundary of strata. Results of Eskin-Okounkov [EO01] allow one to compute these volumes.

Some results in a similar spirit to Theorem 1.1 appear in [Che11] (Appendix A, written with assistance of Alex Eskin) and [CMZ16] (Proposition 17.1). These authors show that certain averages associated to all branched covers of tori with some specified ramification data converge to the (area) Siegel-Veech constant of the entire stratum.

1.6 Outline of the paper.

- Section 2 gives an application to Siegel-Veech constants associated to Veech surfaces in genus 2, an application showing that Siegel-Veech constants are bounded in a fixed stratum, and some results on the set of Siegel-Veech constants associated to all the measures on a stratum.
- Section 3 contains the proof of Theorem 1.1 (Convergence of Siegel-Veech constants), assuming the recurrence-type result Proposition 1.1.
- Section 4 contains the proof of Theorem 1.2 (Uniform quadratic bound for stratum), assuming Proposition 1.1.
- Section 5 presents a self-contained proof of the recurrence-type result Proposition 1.1, following Eskin-Masur ([EM01]), which uses the “system of integral inequalities” approach, and involves technical results on combining “complexes” of saddle connections.
- Section 6 contains a discussion of an upper bound for c_{\max} in Theorem 1.2, which is exponential in the genus g of the surfaces in the stratum.

1.7 Acknowledgements

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2 Applications

2.1 Convergence of Siegel-Veech constants in $\mathcal{H}_1(2)$

We give an application of Theorem 1.1 in genus two, which we then use to give proofs of the convergence of certain sequences of Siegel-Veech constants. The application uses in a crucial way the equidistribution result of Eskin-Mirzakhani-Mohammadi [EMM15].

Let μ_{MV} be the Masur-Veech measure on the stratum $\mathcal{H}_1(2)$.

Theorem 2.1. *Let $\{C_n\}$ be a sequence of distinct closed $SL_2(\mathbb{R})$ -orbits in $\mathcal{H}_1(2)$, and let μ_n be the ergodic $SL_2(\mathbb{R})$ -invariant probability measure whose support is C_n . Then*

$$\lim_{n \rightarrow \infty} c(\mu_n) = c(\mu_{MV}) = \frac{10}{\pi}.$$

Recall that for surfaces generating a closed $SL_2(\mathbb{R})$ orbit (known as Veech surfaces), the generic quadratic growth constant for the whole orbit actually equals the constant for every surface in the orbit ([Vee89], Proposition 3.10). Thus the above result implies that the quadratic growth constants for a sequence of distinct Veech surfaces in $\mathcal{H}_1(2)$ tend to the constant for the whole stratum.

Proof. We claim that $\lim_{n \rightarrow \infty} \mu_n = \mu_{MV}$. By the equidistribution result in [EMM15] (Corollary 2.5), if this were not the case, there would exist a subsequence $k_n \rightarrow \infty$ and \mathcal{N} an affine invariant submanifold of $\mathcal{H}_1(2)$, with each C_{k_n} contained in \mathcal{N} . (An affine invariant manifold is the image of a proper immersion from a connected manifold to a stratum that is cut out locally by homogeneous real linear equations in period coordinates). Now \mathcal{N} cannot be a single closed $SL_2(\mathbb{R})$ -orbit, since it contains infinitely many distinct closed $SL_2(\mathbb{R})$ -orbits. Then by McMullen's classification ([McM07], Theorem 1.2), \mathcal{N} must be the whole stratum $\mathcal{H}_1(2)$, contradiction, establishing the claim.

Now Theorem 1.1 gives $\lim_{n \rightarrow \infty} c(\mu_n) = c(\mu_{MV})$. By [EMZ03] (Example 14.7, second case), $c(\mu_{MV}) = \frac{10}{\pi}$ (the normalization for the Siegel-Veech constant used in that paper differs from ours by a factor of π). ■

The two corollaries below follow immediately from Theorem 2.1.

Corollary 2.1 (Convergence for non-arithmetic Veech surfaces). *Let D be a positive integer that is not a perfect square, with $D \equiv 0, 1 \pmod{4}$, and let E_D be the $SL_2(\mathbb{R})$ -orbit in $\mathcal{H}_1(2)$ of a pair (M, ω) for which the Jacobian $\text{Jac}(M)$ admits real multiplication by \mathcal{O}_D , the ring of integers in $\mathbb{Q}[\sqrt{D}]$, with ω an eigenform (these orbits are known to be closed). Let μ_D be the ergodic $SL_2(\mathbb{R})$ -invariant probability measure whose support is E_D . Then*

$$\lim_{D \rightarrow \infty} c(\mu_D) = c(\mu_{MV}) = \frac{10}{\pi}.$$

Bainbridge found a formula for $c(\mu_D)$ and numerical evidence suggesting that the above convergence holds ([Bai07], discussion after Theorem 14.1).

Corollary 2.2 (Convergence for arithmetic Veech surfaces). *Let $\{S_n\}$ be a sequence of square-tiled surfaces in $\mathcal{H}_1(2)$, where S_n is tiled by exactly k_n squares, with $k_n \rightarrow \infty$. Let μ_n be the ergodic $SL_2(\mathbb{R})$ -invariant probability measure whose support is the (closed) orbit $SL_2(\mathbb{R})S_n$. Then*

$$\lim_{n \rightarrow \infty} c(\mu_n) = c(\mu_{MV}) = \frac{10}{\pi}.$$

This result was conjectured by Lelièvre, who proved it with the additional restriction that each k_n , the number of squares, is a prime, and found numerical evidence for the general case ([Lel06]).

Remark 2.1. One can also use the strategy above to prove that the Siegel-Veech constants corresponding to the eigenform loci in $\mathcal{H}(1, 1)$ (which are no longer just closed orbits) converge to the Siegel-Veech constant for $\mathcal{H}(1, 1)$. For non-arithmetic eigenform loci, this was proven by Bainbridge; in fact the Siegel-Veech constants are the same for all the non-arithmetic eigenform loci ([Bai10], Theorem 1.5). For the arithmetic eigenform loci, convergence was proven by Eskin-Masur-Schmoll ([EMS03], Theorem 1.3); here the sequence of Siegel-Veech constants is not eventually constant.

2.2 Boundedness of Siegel-Veech constants

Theorem 2.2. *Fix a stratum \mathcal{H}_1 . There exists a bound B (depending on the stratum) such that for any ergodic $SL_2(\mathbb{R})$ -invariant probability measure μ on \mathcal{H}_1 ,*

$$c(\mu) \leq B.$$

We give two different proofs.

Proof via Theorem 1.1. Suppose, for the sake of contradiction, that μ_n is a sequence of ergodic $SL_2(\mathbb{R})$ -invariant probability measures on \mathcal{H}_1 , with $c(\mu_n) \rightarrow \infty$. By passing to a subsequence, and applying the equidistribution theorem [EMM15] (Corollary 2.5), we can assume that $\mu_n \rightarrow \eta$, where η is another ergodic $SL_2(\mathbb{R})$ -invariant probability measure. Then Theorem 1.1 gives that

$$\lim_{n \rightarrow \infty} c(\mu_n) = c(\eta) < \infty,$$

contradicting our assumption. ■

Proof via Theorem 1.2. We claim that for any such μ , $c(\mu) \leq c_{\max}$, where c_{\max} is the constant in Theorem 1.2. By [EM01], there exists some $X \in \mathcal{H}_1$ (in fact the following will hold for μ -a.e. X) such that

$$N(X, R) \sim c(\mu)R^2.$$

Hence by Theorem 1.2, $c(\mu) \leq c_{\max}$. ■

2.3 The set of Siegel-Veech constants

Using Theorem 1.1, we can easily prove several results showing that the set of Siegel-Veech constants of all the measures for a fixed stratum is not too complicated.

Theorem 2.3. *Fix a stratum \mathcal{H} . Let*

$$S(\mathcal{H}) = \{c(\mu) : \mu \text{ an ergodic } SL_2(\mathbb{R})\text{-invariant probability measure on } \mathcal{H}_1\}.$$

Then $S(\mathcal{H})$ is closed as a subset of \mathbb{R} .

This will follow as the $n = 0$, $\mathcal{N} = \mathcal{H}$ case of Proposition 2.1 below.

By Eskin-Mirzakhani-Mohammadi [EMM15], the ergodic $SL_2(\mathbb{R})$ -invariant probability measures on \mathcal{H} are in bijection with the set of affine invariant submanifolds \mathcal{M} of \mathcal{H} . We define $c(\mathcal{M})$ to equal $c(\mu)$, where μ is the measure corresponding to \mathcal{M} .

Proposition 2.1. *Fix an affine invariant submanifold \mathcal{N} of \mathcal{H} .*

$$S_n(\mathcal{N}) = \{c(\mathcal{M}) : \mathcal{M} \subset \mathcal{N}, \dim_{\mathbb{C}} \mathcal{M} \geq n\}.$$

Then $S_n(\mathcal{N})$ is closed as a subset of \mathbb{R} .

Proof. Suppose $x \in \mathbb{R}$ and $x = \lim_{k \rightarrow \infty} c(\mathcal{M}_k)$ for some sequence \mathcal{M}_k of affine invariant submanifolds, corresponding to measures μ_k . Consider the set of all affine invariant submanifolds that contain infinitely many of the \mathcal{M}_k , and pick an element \mathcal{M} that is minimal (with respect to inclusion) in this set. The set is non-empty (since \mathcal{N} is in it), and a minimal element exists because the longest chain (with respect to inclusion) has cardinality at most $\dim_{\mathbb{C}}(\mathcal{N}) < \infty$. Note that $\dim_{\mathbb{C}} \mathcal{M} \geq n$, hence $c(\mathcal{M}) \in S_n(\mathcal{N})$. Now by equidistribution ([EMM15], Corollary 2.5), we can find a subsequence j_k such that μ_{j_k} converges to μ , where μ is the measure corresponding to \mathcal{M} .

By Theorem 1.1, $c(\mathcal{M}) = c(\mu) = \lim_{k \rightarrow \infty} c(\mu_{j_k}) = \lim_{k \rightarrow \infty} c(\mathcal{M}_{j_k}) = x$. Hence $x \in S_n(\mathcal{N})$, and we are done. ■

Given a closed $X \subset \mathbb{R}$, we define the *derived set* X^* to be the set obtained from X by removing all the isolated points. We let $X^{*n} = (\cdots (X^*)^* \cdots)^*$, where there are n occurrences of $*$. We define the *rank*, $\text{rank}(X)$, to be the smallest n for which $X^{*n} = \{\}$; if no such n exists, we declare the rank to be infinity. (Note that this is a slight variation of the usual notion of *Cantor-Bendixson rank*.)

Theorem 2.4. *Let $\mathcal{N} \subset \mathcal{H}$ be an affine invariant submanifold with $\dim_{\mathbb{C}} \mathcal{N} = d$. Then*

$$\text{rank } S_n(\mathcal{N}) \leq d - n + 1.$$

In particular, $\text{rank } S(\mathcal{H}) \leq \dim_{\mathbb{C}} \mathcal{H} + 1$.

Proof. We argue by downwards induction on n .

The base case is $n = d$. Here the only affine invariant submanifold of \mathcal{N} with dimension at least d is \mathcal{N} itself. Thus $S_d(\mathcal{N}) = \{c(\mathcal{N})\}$, which has rank 1, so this gives the base case.

For the inductive step assume the result for n .

We claim that $S_{n-1}(\mathcal{N})^* \subset S_n(\mathcal{N})$. To see this, let $x \in S_{n-1}(\mathcal{N})^*$, which means we can find a sequence of distinct \mathcal{M}_i of dimension at least $n - 1$ with $\lim_i c(\mathcal{M}_i) = x$. As in the proof of Proposition 2.1, using [EMM15] (Corollary 2.5) we can find \mathcal{N} containing all \mathcal{M}_{j_i} for some subsequence j_i , and $c(\mathcal{N}) = \lim_i c(\mathcal{M}_{j_i}) = x$. Since \mathcal{N} contains distinct manifolds of dimension at least $n - 1$, it must have dimension at least n (this uses the fact that affine invariant submanifolds come from *proper* immersions). So $x = c(\mathcal{N}) \in S_n(\mathcal{N})$. This completes the proof of the claim.

Now for any closed sets $A \subset B \subset \mathbb{R}$, it follows immediately from our definition that $\text{rank}(A) \leq \text{rank}(B)$. Then, from the definition of rank, this fact, the claim above, and the inductive assumption,

$$\text{rank}(S_{n-1}(\mathcal{N})) \leq \text{rank}(S_{n-1}(\mathcal{N})^*) + 1 \leq \text{rank}(S_n(\mathcal{N})) + 1 \leq (d - n + 1) + 1 = d - (n - 1) + 1,$$

which completes the induction. ■

3 Proof of Theorem 1.1 assuming Proposition 1.1

Proof of Theorem 1.1. The proof uses techniques from [EM01] to bound certain integrals defining the Siegel-Veech constants.

The Siegel-Veech constant can be defined by

$$c(\mu) = \frac{\int_{\mathcal{H}_1} \hat{f} d\mu}{\int_{\mathbb{R}^2} f d\lambda},$$

for any $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ continuous and compactly supported. Here $\hat{f} : \mathcal{H}_1 \rightarrow \mathbb{R}$ is the Siegel-Veech transform of f given by

$$\hat{f}(X) := \sum_{c \in \Lambda(X)} f(c),$$

where $\Lambda(X)$ is the multi-set given by taking all holonomies (which gives an element of \mathbb{R}^2) of cylinders on X .

Hence to prove Theorem 1.1, it suffices to show that

$$\lim_{n \rightarrow \infty} \int_{\mathcal{H}_1} \hat{f} d\mu_n = \int_{\mathcal{H}_1} \hat{f} d\eta$$

for all such f . Note that if \hat{f} were compactly supported, this would follow immediately from the definition of weak-* convergence. The idea is to approximate \hat{f} by compactly supported functions, and bound the integral of the error term using integrability results from [EM01]. The key point is that we need a bound for the error term that is independent of the particular measure μ .

Let $C_K = \{X \in \mathcal{H}_1 : \frac{1}{\ell(X)} \leq K\}$. These sets are compact. Now let χ_K be a continuous function $\mathcal{H}_1 \rightarrow [0, 1]$ whose value is 1 on C_K and 0 on $\mathcal{H}_1 \setminus C_{K+1}$. Define $\hat{f}_K = \hat{f} \cdot \chi_K$. Note that \hat{f}_K is compactly supported, hence

$$\lim_{n \rightarrow \infty} \int_{\mathcal{H}_1} \hat{f}_K d\mu_n = \int_{\mathcal{H}_1} \hat{f}_K d\eta.$$

It remains to show that by choosing K large, we can make $\left| \int_{\mathcal{H}_1} \hat{f} d\mu - \int_{\mathcal{H}_1} \hat{f}_K d\mu \right|$ uniformly small for any choice of ergodic $SL_2(\mathbb{R})$ -invariant probability measure μ . We can assume f , and hence \hat{f} , are non-negative, since we only need to show the equality for some f for which $\int_{\mathcal{H}_1} \hat{f} d\eta$ is non-zero. Since f is compactly supported, it is dominated by some multiple of an indicator function of a large ball. By [EM01] Theorem 5.1(a), it follows that $\hat{f} < C/\ell^{1+\delta}$ for some C and $0 < \delta < 1/2$ (the cited result is about counts of saddle connections, but since every cylinder is bounded by saddle connections, we get the same bound for cylinders). Then

$$\left| \int_{\mathcal{H}_1} \hat{f} d\mu - \int_{\mathcal{H}_1} \hat{f}_K d\mu \right| \leq \int_{\mathcal{H}_1 \setminus C_K} \hat{f} d\mu \leq C \int_{\mathcal{H}_1 \setminus C_K} \frac{1}{\ell^{1+\delta}} d\mu \quad (1)$$

$$\leq C \frac{1}{K^{\delta'}} \int_{\mathcal{H}_1 \setminus C_K} \frac{1}{\ell^{1+\delta+\delta'}} d\mu, \quad (2)$$

where we choose $\delta' > 0$ such that $\delta + \delta' < 1/2$. We introduce this extra δ' to get the $\frac{1}{K^{\delta'}}$ term in the above, which will allow us to get decay as $K \rightarrow \infty$. By [EM01] Lemma 5.5, the last integral above is finite; we need to show the somewhat stronger statement that it is bounded from above independent of the choice of μ .

The idea is to replace the integral of $\frac{1}{\ell^{1+\delta+\delta'}}$ over the whole stratum by the integral over large circles centered at a μ -generic point, using Nevo's theorem, and then use Proposition 1.1 to bound the integrals over circles. To apply Nevo's theorem, we need to choose a smoothing function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ that is non-negative, smooth, and compactly supported (having the smoothing is fine for our purposes; Nevo's result should also be true without having to smooth).

Now by Nevo's theorem (see [EM01] Theorem 1.5) and Proposition 1.1, for μ a.e. $X \in \mathcal{H}_1$,

$$\begin{aligned} \int_{\mathcal{H}_1} \frac{1}{\ell^{1+\delta+\delta'}} d\mu \cdot \int_{-\infty}^{\infty} \phi(t) dt &= \lim_{\tau \rightarrow \infty} \int_{-\infty}^{\infty} \phi(\tau - t) \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\ell(g_t r_\theta X)^{1+\delta+\delta'}} d\theta \right) dt \\ &\leq b \int_{-\infty}^{\infty} \phi(t) dt. \end{aligned}$$

Hence $\int_{\mathcal{H}_1} \frac{1}{\ell^{1+\delta+\delta'}} d\mu \leq b$, for any μ . Plugging into (2) gives, for all μ ,

$$\left| \int_{\mathcal{H}_1} \hat{f} d\mu - \int_{\mathcal{H}_1} \hat{f}_K d\mu \right| \leq C \cdot \frac{b}{K^{\delta'}}. \quad (3)$$

Now we put everything together. Fix $\epsilon > 0$. Choose K large so that $C \cdot b/K^{\delta'} < \epsilon/3$. Now choose N such that

$$\left| \int_{\mathcal{H}_1} \hat{f}_K d\mu_n - \int_{\mathcal{H}_1} \hat{f}_K d\eta \right| < \epsilon/3 \quad (4)$$

for all $n \geq N$.

Then by the triangle inequality, and (3), (4), for $n \geq N$,

$$\begin{aligned} \left| \int_{\mathcal{H}_1} \hat{f} d\mu_n - \int_{\mathcal{H}_1} \hat{f} d\eta \right| &\leq \left| \int_{\mathcal{H}_1} \hat{f} d\mu_n - \int_{\mathcal{H}_1} \hat{f}_K d\mu_n \right| + \left| \int_{\mathcal{H}_1} \hat{f}_K d\mu_n - \int_{\mathcal{H}_1} \hat{f}_K d\eta \right| + \left| \int_{\mathcal{H}_1} \hat{f}_K d\eta - \int_{\mathcal{H}_1} \hat{f} d\eta \right| \\ &\leq C \cdot b/K^{\delta'} + \epsilon/3 + C \cdot b/K^{\delta'} < 3(\epsilon/3) = \epsilon, \end{aligned}$$

and we are done. ■

Remark 3.1. Some similar ideas appear in the proof of Theorem 2.4 in [EMS03].

4 Proof of Theorem 1.2 assuming Proposition 1.1

Proof of Theorem 1.2. The proof is just a slight modification of the Eskin-Masur proof of Masur's original (non-uniform) quadratic upper bound (Theorem 5.4 in [EM01]; originally proved in [Mas90]).

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the indicator function of the trapezoid with vertices $(\pm 1, 1), (\pm 1/2, 1/2)$, and let $\hat{f} : \mathcal{H}_1 \rightarrow \mathbb{R}$ be its Siegel-Veech transform. The Eskin-Masur strategy counts cylinders with holonomy in a ball by applying g_T to the trapezoid above, making it long and thin, and then rotating it around using r_θ (this actually counts cylinders whose length lies in $[R/2, R]$). This technique gives the first inequality below; the second comes from the fact that the trapezoid is contained in the ball of radius 2; the third uses Theorem 5.1(a) in [EM01]; and the fourth comes from applying Proposition 1.1:

$$\begin{aligned} N(X, R) - N(X, R/2) &\leq R^2 \int_0^{2\pi} \hat{f}(g_{\log R} r_\theta X) d\theta \\ &\leq R^2 \int_0^{2\pi} N(g_{\log R} r_\theta X, 2) d\theta \\ &\leq R^2 \int_0^{2\pi} \frac{c_1 d\theta}{\ell(g_{\log R} r_\theta X)^{1+\delta}} \\ &\leq c_1 R^2 \left(c_0 e^{-(\log R)(1-2\delta)} \alpha(X) + b \right). \end{aligned}$$

The constants c_1, c_0, b do not depend on X or R . For R large (depending on X), we can make $c_2 e^{-(\log R)(1-2\delta)} \alpha(X) < b$, and so we get

$$N(X, R) - N(X, R/2) \leq 2bc_1 R^2.$$

A straight-forward geometric series argument then gives the desired inequality.

We also see that the function $R_0(X)$ can be chosen to depend only on $\alpha(X)$ (in an explicit way). The function $\alpha(X)$ is itself bounded by an explicit function of $\ell(X)$ (and the genus of the stratum). \blacksquare

5 Proof of Proposition 1.1

5.1 Outline of proof

The proof of the key technical tool Proposition 1.1 follows the “system of integral inequalities” approach. This argument comes up when one is trying to prove bounds on integrals of certain functions over either $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$, the moduli space of lattices, or \mathcal{H}_1 , a connected component of a stratum of the moduli space of unit-area translation surfaces. The argument was first used in the lattice context, where it is the key technical tool for the upper bound in the proof of quantitative Oppenheim ([EMM98]). The function to be integrated is dominated by a multiple of $\frac{1}{\ell}$, where ℓ is the length of the shortest vector in a lattice, or the shortest saddle connection on a translation surface. Thus it suffices to bound $\int \frac{1}{\ell}$.

Here is a sketch of the proof of Proposition 1.1 (following [EM01]) and an outline of the rest of the paper. We will end up proving the more general Proposition 5.3.

- Suppose (unrealistically) that the shortest saddle connection on each $g_T r_\theta X$ is just the image of the shortest saddle connection s on X under $g_T r_\theta$. A straightforward $SL_2(\mathbb{R})$ calculation gives that in this case we would actually get exponential decay in T of the integral of $1/\ell$ over the circle of radius T . This calculation is done in Section 5.2.

- We will then use a pointwise argument to take care of the case in which the shortest vector on $g_T r_\theta X$ comes from some other s' on X . The idea is to combine s, s' into a complex whose boundary consists of short saddle connections. The necessary facts about combining complexes are proved in Section 5.3.
- Our goal is then prove a generalization of the desired theorem with $1/\ell$ replaced by α_k , which is defined to be the smallest y such that all complexes of complexity k (and some upper bound on area) have a saddle connection of length at least y . This generalization is stated at the end of Section 5.3.
- In Section 5.4 we prove a bound for the integral of α_k over a circle of fixed radius τ in terms of the values of α_j , for $j \geq k$, at the center of the circle. This uses the pointwise argument that involves combining complexes. Unfortunately, there are large terms in the inequality that involve τ .
- To get around this dependence on the radius τ , in Section 5.5, we move out in many steps of size τ to get to a large circle of radius T , so we can then think of τ as some constant. This involves some hyperbolic geometry estimates.
- Finally, we put everything together in Section 5.6. This will involve downwards induction on k , so that we can deal with the higher complexity error terms that pop up.

5.2 Decay for a single vector

In this section we fix a saddle connection s on X and consider the length of the corresponding saddle connection on $g_t r_\theta X$. This is really just a question about $SL_2(\mathbb{R})$.

Proposition 5.1. *Fix $0 \leq \delta < 1$, and let v be a vector in \mathbb{R}^2 . Then*

$$\int_0^{2\pi} \frac{1}{\|g_t r_\theta v\|^{1+\delta}} d\theta \leq c(\delta) \frac{e^{-t(1-\delta)}}{\|v\|^{1+\delta}},$$

for all $t \geq 0$, where $c(\delta)$ is a constant depending only on δ .

Proof. By rotating and scaling, we can assume that $v = (1, 0)$. Also, by symmetry, it suffices to consider $[-\pi/2, \pi/2]$ instead of $[0, 2\pi]$ as the domain of integration. Then

$$\int_{-\pi/2}^{\pi/2} \frac{1}{\|g_t r_\theta v\|^{1+\delta}} d\theta = \int_{-\pi/2}^{\pi/2} (e^{-2t} \cos^2 \theta + e^{2t} \sin^2 \theta)^{-(1+\delta)/2} d\theta.$$

We now divide $[-\pi/2, \pi/2]$ into two pieces:

$$S_1 = \{\theta : e^{-2t} \cos^2 \theta > e^{2t} \sin^2 \theta\}$$

$$S_2 = \{\theta : e^{-2t} \cos^2 \theta < e^{2t} \sin^2 \theta\}.$$

Note that $\theta \in S_1$ iff θ is very close to 0, and for such angles $\sin \theta \approx \theta$ and $\cos \theta \approx 1$. It follows that there are constants c_1, c_2 such that

$$[-c_1 e^{-2t}, c_1 e^{-2t}] \subset S_1 \subset [-c_2 e^{-2t}, c_2 e^{-2t}].$$

Now

$$\begin{aligned} \int_{S_1} (e^{-2t} \cos^2 \theta + e^{2t} \sin^2 \theta)^{-(1+\delta)/2} d\theta &\leq \int_{S_1} (e^{-2t} \cos^2 \theta)^{-(1+\delta)/2} d\theta \\ &= O\left(e^{(1+\delta)t} |S_1|\right) \\ &= O\left(e^{(1+\delta)t} (2c_2 e^{-2t})\right) \\ &= O(e^{-t(1-\delta)}). \end{aligned}$$

It remains to prove a similar bound for the other part of the domain of integration.

$$\begin{aligned}
\int_{S_2} (e^{-2t} \cos^2 \theta + e^{2t} \sin^2 \theta)^{-(1+\delta)/2} d\theta &\leq \int_{S_2} (e^{2t} \sin^2 \theta)^{-(1+\delta)/2} d\theta \\
&= e^{-t(1+\delta)} \int_{S_2} |\sin \theta|^{-(1+\delta)} d\theta \\
&\leq c \cdot e^{-t(1+\delta)} \int_{S_2} |\theta|^{-(1+\delta)} d\theta \\
&\leq 2c \cdot e^{-t(1+\delta)} \int_{c_1 e^{-2t}}^{\pi/2} |\theta|^{-(1+\delta)} d\theta \\
&\leq c' \cdot e^{-t(1+\delta)} [-|\theta|^{-\delta}]_{c_1 e^{-2t}}^{\pi/2} \\
&= c'' \cdot e^{-t(1-\delta)}.
\end{aligned}$$

Putting together the two bounds yields the desired result. ■

5.3 Complexes of saddle connections

In this section we study certain collections of saddle connections called complexes. There is no bound over the whole stratum on the number of saddle connections on X shorter than some specified ϵ , since one can find “small” subsurfaces (eg a small cylinder) which contain lots of short saddle connections. To get around this, we build complexes, which have a notion of complexity that cannot increase indefinitely.

Definition 5.1. A *complex* K in X is a closed subset of X whose boundary ∂K consists of a union of disjoint saddle connections (when we say that two saddle connections are disjoint, we mean that the interiors are disjoint), such that if ∂K contains three saddle connections that bound a triangle, then the interior of that triangle is in K . We will denote by $|\partial K|$ the length of the longest saddle connection in ∂K .

Definition 5.2. Given a complex K , the *complexity* of K is the number of saddle connections needed to triangulate K (a triangulation of a complex K is a collection S of saddle connections together with a collection of triangles T , with disjoint interiors, each of whose boundaries consist of elements of S , such that $K = S \cup T$).

Note, by an Euler characteristic argument, the number of saddle connections in any triangulation of K is independent of the triangulation.

Informally, Proposition 5.2 below says that if we have a complex whose boundary consists of short saddle connections, and we find a short saddle connection that is either disjoint from the complex or crosses the boundary, then we can extend the complex to a complex of higher complexity whose boundary saddle connections are still short.

Proposition 5.2. *Suppose K is complex of complexity k with non-empty boundary, and let σ be a saddle connection which is either disjoint from K or crosses ∂K . Then there is a complex $K' \supset K$ of complexity $i > k$ such that*

$$(i) \quad |\partial K'| \leq |\partial K| + \max(|\partial K|, |\sigma|),$$

$$(ii) \quad \text{area}(K') \leq \text{area}(K) + |\partial K| \max(|\partial K|, |\sigma|).$$

Proof sketch (see [EM01] for full details): If σ is disjoint from K , then we just take $K' = K \cup \sigma$ which has complexity $k + 1$ and the length of the boundary clearly satisfies the desired inequality (i). The area does not increase, so (ii) is also satisfied.

If σ crosses ∂K , there are various cases to work out.

Consider first the case where σ has one endpoint in K and one outside of K , and $\sigma \cap K$ is connected. Let s be the saddle connection in ∂K that σ crosses. Let a, b be endpoints of s , and let $q = s \cap \sigma$, and let p be the endpoint of σ that lies outside of K . Let l_a be the path that runs from a to q along s , and then from q to p along σ , and define l_b similarly. In the homotopy class of l_a (rel endpoints), there is a geodesic segment g_b , which consists of a union of saddle connections, and we define g_b analogously. If at least one of g_a, g_b has a saddle connection not contained in K , then we add that saddle connection to K to form K' . Otherwise, the region bounded by s, g_a, g_b is not a triangle, and we can choose some singularity p' on WLOG g_a not contained in K , and then add to K the shortest saddle connection connecting b to p' . Whatever happens, both (i) and (ii) are satisfied.

The other case is similar. ■

While we are primarily interested in studying the function $1/\ell(X)$, where $\ell(X)$ is the length of the shortest saddle connection on X , we will be forced to also consider complexes in which all the boundary saddle connections are short.

Fix $\delta > 0$. Let $\beta = \frac{1}{M+1}$, where M is the complexity of X . We define a sequence of functions

$$\alpha_i(X) = \max_K \frac{1}{|\partial K|^{1+\delta}},$$

where the max is taken over

$$\{K : K \text{ complex in } X \text{ of complexity } i, \text{ area}(K) < i\beta\}. \quad (5)$$

For some i, X , if there are no complexes satisfying (5), then we set $\alpha_i(X) = 0$. We need the area restriction to keep the complexes from getting too big - in particular, we need to avoid considering a complex that is equal to the whole surface. Note that $\alpha_1(X) = 1/\ell(X)^{1+\delta}$, since a complex of complexity 1 must be a single saddle connection, and this has zero area.

The following theorem is the generalization of Proposition 1.1 needed for the downwards induction on the complexity k . The proof will be completed in Section 5.6.

Proposition 5.3. *For any stratum \mathcal{H}_1 , and $0 < \delta < 1/2$, we can find constants c, b such that for any k and $X \in \mathcal{H}_1$,*

$$\int_0^{2\pi} \alpha_k(g_T r_\theta X) d\theta < c_0 \cdot e^{-(1-2\delta)T} \sum_{j \geq k} \alpha_j(X) + b,$$

for all $T \geq 0$.

Proof of Proposition 1.1 assuming Proposition 5.3: Apply Proposition 5.3 with $k = 1$. Note that $\alpha_1(X) = \frac{1}{\ell(X)^{1+\delta}}$. Defining $\alpha(X) := \sum_{j \geq k} \alpha_j(X)$ gives the desired statement. ■

5.4 Averaging over a circle of bounded size

Given a function f on \mathbb{H} and a point $X \in \mathbb{H}$, we let

$$\text{Ave}_t(f)(X) := \frac{1}{2\pi} \int_0^{2\pi} f(g_t r_\theta X) d\theta.$$

Proposition 5.4. *Fix \mathcal{H}_1 and $0 < \delta < 1$. There exists $C > 0$, such that for any $t > 0$, there exist constants b_t, w_t such that for any k and $X \in \mathcal{H}_1$,*

$$\text{Ave}_t(\alpha_k)(X) \leq C e^{-t(1-\delta)} \alpha_k(X) + w_t \sum_{j > k} \alpha_j(X) + b_t.$$

Proof. Let K be a complex on X of complexity k realizing the definition of $\alpha_k(X)$, and let $K' = K'(\theta)$ be a complex on $g_t r_\theta X$ realizing $\alpha_k(g_t r_\theta X)$.

Here is the idea of the proof. The first term on the right hand side of the bound comes from the case when $K' = g_t r_\theta(K)$; in this case we get a bound from Proposition 5.1. The second term comes from the case where K' is some other complex; in this case we get a bound by combining $(g_t r_\theta)^{-1}K'$ with K to get a higher complexity complex. For this we assume that ∂K consists of short saddle connections. The third term handles the case when none of the saddle connections in K are short.

Let $E \subset [0, 2\pi)$ be the set of θ for which $K' = g_t r_\theta K$, and let F be the complement. Then

$$\text{Ave}_t(\alpha_k)(X) = \frac{1}{2\pi} \left(\int_E \alpha_k(g_t r_\theta X) d\theta + \int_F \alpha_k(g_t r_\theta X) d\theta \right).$$

1. To bound the integral over E we apply Proposition 5.1:

$$\int_E \alpha_k(g_t r_\theta X) d\theta = \int_E \min_{s \in \partial K} \frac{1}{\|g_t r_\theta s\|^{1+\delta}} d\theta \quad (6)$$

$$\leq \min_{s \in \partial K} \int_0^{2\pi} \frac{1}{\|g_t r_\theta s\|^{1+\delta}} d\theta \quad (7)$$

$$\leq \min_{s \in \partial K} c(\delta) \frac{e^{-t(1-\delta)}}{\|s\|^{1+\delta}} = c(\delta) e^{-t(1-\delta)} \alpha_k(X). \quad (8)$$

2. Now we bound the integral over F in a pointwise fashion. First assume that

$$|\partial K| \geq e^{-2t} \sqrt{\beta}.$$

Then we get the bound

$$\int_F \alpha_k(g_t r_\theta X) d\theta \leq \int_0^{2\pi} \alpha_k(g_t r_\theta X) d\theta \leq \int_0^{2\pi} (e^t)^{1+\delta} \alpha_k(X) d\theta \quad (9)$$

$$\leq 2\pi (e^t)^{1+\delta} (e^{-2t} \sqrt{\beta})^{-(1+\delta)} =: b_t. \quad (10)$$

3. Now assume that

$$|\partial K| \leq e^{-2t} \sqrt{\beta}. \quad (11)$$

Note that because of our assumption (5) on the area of the complexes that we allow in the definition of α_k , we know that neither K, K' is all of the surface, and hence each must have non-empty boundary. It follows that some saddle connection \tilde{s} in $\partial(g_t r_\theta)^{-1}K'$ must either be disjoint from K , or cross ∂K . By Proposition 5.2, from \tilde{s} and K we can form a new complex \tilde{K} on X with complexity $i > k$ such that

$$\begin{aligned} |\partial \tilde{K}| &\leq |\partial K| + \max(|\partial K|, |\tilde{s}|) \leq |\partial K| + \max(|\partial K|, |\partial(g_t r_\theta)^{-1}K'|) \\ &\leq 2|\partial(g_t r_\theta)^{-1}K'| \leq 2e^t |\partial K'|, \end{aligned}$$

and

$$\text{area}(\tilde{K}) \leq \text{area}(K) + |\partial K| \max(|\partial K|, |\tilde{s}|). \quad (12)$$

Now K' realizes the maximum in the definition of $\alpha_k(g_t r_\theta X)$, so in particular

$$|\partial K'| \leq |\partial(g_t r_\theta K)| \leq e^t |\partial K| \leq e^t (e^{-2t} \sqrt{\beta}) = e^{-t} \sqrt{\beta}.$$

Thus $|\tilde{s}| \leq e^t |\partial K'| \leq \sqrt{\beta}$. Starting with the inequality (12), and using bound on area from (5) together with our assumption (11) on $|\partial K|$, we get

$$\begin{aligned} \text{area}(\tilde{K}) &\leq k\beta + (e^{-2t}\sqrt{\beta}) \max(e^{-2t}\sqrt{\beta}, \sqrt{\beta}) \\ &\leq k\beta + \beta \leq i\beta. \end{aligned}$$

So the complex \tilde{K} is one of those over which the maximum in the definition of $\alpha_i(X)$ is taken. Also, since $\beta = \frac{1}{M+1}$, we have $\text{area}(\tilde{K}) < 1$, and hence $\partial\tilde{K}$ must be non-empty. It follows that

$$\alpha_i(X) \geq \frac{1}{|\partial\tilde{K}|^{1+\delta}} \geq \frac{1}{(2e^t)^{1+\delta}} \frac{1}{|\partial K'|^{1+\delta}} = \frac{1}{(2e^t)^{1+\delta}} \alpha_k(g_t r_\theta(X)).$$

So

$$\alpha_k(g_t r_\theta(X)) \leq (2e^t)^{1+\delta} \alpha_i(X).$$

Using this pointwise bound we get (under the assumption (11)) that

$$\int_F \alpha_k(g_t r_\theta X) d\theta \leq \int_F (2e^t)^{1+\delta} \sum_{i>k} \alpha_i(X) \leq 2\pi (2e^t)^{1+\delta} \sum_{i>k} \alpha_i(X). \quad (13)$$

Now we put the three parts together. By (8), (10), and (13), we have

$$\text{Ave}_t(\alpha_k)(X) \leq \frac{c(\delta)}{2\pi} e^{-t(1-\delta)} \alpha_k(X) + (2e^t)^{1+\delta} \sum_{i>k} \alpha_i(X) + \frac{b_t}{2\pi}.$$

■

5.5 Averaging over larger circles

To prove Proposition 5.3, we need to compare the average of the function α_i over a large circle to the value of α_i at the center of the circle. Proposition 5.4 gives a comparison, but the w_t term means as we make t large we lose control over the size of the average. To get around this, we move in steps, repeatedly applying Proposition 5.4 over circles of some fixed size τ . To go from a circle of radius t to one of radius $t + \tau$, we need Lemma 5.2 (Shadowing) below.

We first need a preliminary hyperbolic geometry lemma. Let us choose $\kappa > 0$ such that for any $X, Y \in \mathcal{H}_1$ that are in the same $SL_2(\mathbb{R})$ orbit, and satisfy $d(X, Y) < \kappa$, we have $\frac{1}{2}\alpha_i(X) \leq \alpha_i(Y) \leq 2\alpha_i(X)$. Here $d(X, Y)$ is defined as the Teichmüller distance between the projections of X, Y to the moduli space. Such a choice of κ is possible because the Teichmüller distance of the projections controls the ratio of lengths of corresponding saddle connections on the surfaces.

Now let $U = U(t, \tau, \kappa) := \{\phi : d(g_{\tau} r_\phi g_t X, X) \geq t + \tau - \kappa\}$. In Figure 1, U is the set of angles ϕ corresponding to the dashed segment.

Lemma 5.1. *There exists $c' > 0$ such that for all t, τ we have $|U(t, \tau, \kappa)| \geq c'$.*

The lemma says that, in the hyperbolic plane, if we consider a circle of radius t centered at some point p , then at least a definite, positive proportion of any circle of radius τ centered at a point on the first circle will lie outside the disk of radius $t + \tau - \kappa$ centered at p . The proportion depends on κ , but not on t or τ .

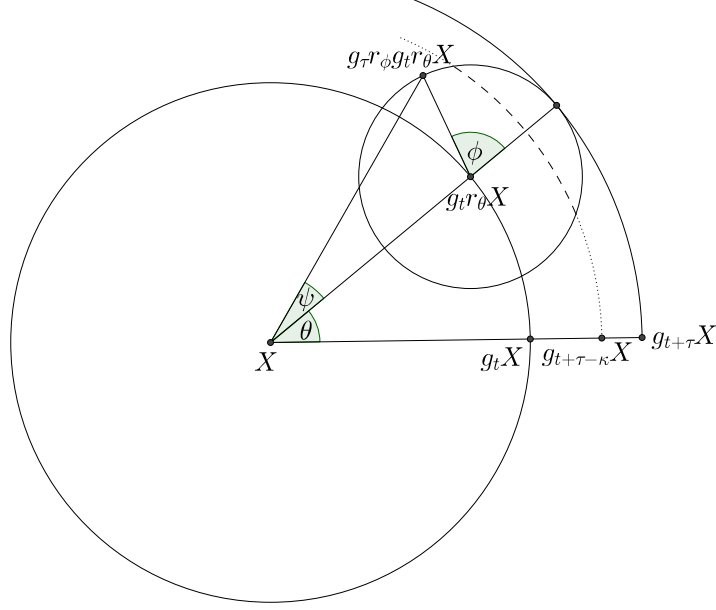


Figure 1: Comparing averages for Lemma 5.2 (Shadowing)

Proof. Consider the triangle with vertices X , $Y = g_t X$, and $Z = g_{\tau} r_{\phi} g_t X$. Note that $|XY| = t$, $|YZ| = \tau$, and $\angle XYZ = \pi - \phi$. Applying the Hyperbolic Law of Cosines to the triangle XYZ gives

$$\cosh(XZ) = \cosh(t) \cosh(\tau) + \sinh(t) \sinh(\tau) \cos \phi.$$

Now for large x , we have that $\cosh(x) \approx \frac{e^x}{2} \approx \sinh(x)$. Using this approximation we get that

$$\frac{e^{|XZ|}}{2} \approx \left(\frac{e^t}{2} \frac{e^{\tau}}{2} \right) (1 + \cos \phi),$$

so

$$|XZ| \approx t + \tau + \log \frac{1 + \cos \phi}{2}.$$

Now $\phi \in U$ iff $|XZ| \geq t + \tau - \kappa$. If the above approximation is accurate enough, we get that $XZ \geq t + \tau - \kappa$, when $|\phi| \leq c'$, for some c' independent of t, τ , as desired. To justify the approximation, we can absorb the error in a multiplicative term that is close to 1 when t, τ are large, and this just makes c' smaller by a multiplicative factor. To handle bounded t, τ , we make the constant c' smaller if necessary. \blacksquare

Lemma 5.2 (Shadowing). *There exists a constant $c_2 > 0$ such that for any $t, \tau \geq 0$ and $X \in \mathcal{H}_1$,*

$$\text{Ave}_{t+\tau}(\alpha_i)(X) \leq c_2 \int_0^{2\pi} \text{Ave}_{\tau}(\alpha_i)(g_t r_{\theta} X) d\theta.$$

Proof. We want to parametrize the point $g_\tau r_\phi g_t r_\theta X$ by the angle ψ as indicated in the diagram, i.e. we want $g_\tau r_\phi g_t r_\theta X = g_s r_{\theta+\psi} X$, where s, ψ are functions of t, τ, ϕ . Let $\Psi : S^1 \rightarrow S^1$ be the map taking ϕ to ψ . On small intervals, this map is a diffeomorphism to its image. Now changing the variable from ϕ to ψ , and using the defining property of κ , we get

$$\begin{aligned} \text{Ave}_\tau(\alpha_i)(g_t r_\theta X) &\geq \frac{1}{2\pi} \int_U \alpha_i(g_\tau r_\phi g_t r_\theta X) d\phi \\ &= \frac{1}{2\pi} \int_{\Psi(U)} \alpha_i(g_\tau r_{\phi(\psi)} g_t r_\theta X) \left| \frac{d\phi}{d\psi} \right| d\psi \\ &\geq \frac{1}{2\pi} \int_{\Psi(U)} \frac{1}{2} \alpha_i(g_{t+\tau} r_{\theta+\psi} X) \left| \frac{d\phi}{d\psi} \right| d\psi. \end{aligned}$$

Now, to estimate the right hand term in the statement of the lemma, we let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and perform the linear change of variables

$$\begin{pmatrix} \theta \\ \psi \end{pmatrix} \mapsto \begin{pmatrix} \xi \\ \psi \end{pmatrix} = A \begin{pmatrix} \theta \\ \psi \end{pmatrix} = \begin{pmatrix} \theta + \psi \\ \psi \end{pmatrix}.$$

where the Jacobian of A is 1. Integrating the inequality above over θ and performing this change of variables gives

$$\begin{aligned} \int_0^{2\pi} \text{Ave}_\tau(\alpha_i)(g_t r_\theta X) d\theta &\geq \int_0^{2\pi} \frac{1}{4\pi} \int_{\Psi(U)} \alpha_i(g_{t+\tau} r_{\theta+\psi} X) \left| \frac{d\phi}{d\psi} \right| d\psi d\theta \\ &= \frac{1}{4\pi} \int \int_{A([0, 2\pi] \times \Psi(U))} \alpha_i(g_{t+\tau} r_\xi X) \left| \frac{d\phi}{d\psi}(\psi) \right| d\xi d\psi \\ &= \frac{1}{4\pi} \int_{\Psi(U)} \int_{\psi+[0, 2\pi]} \alpha_i(g_{t+\tau} r_\xi X) \left| \frac{d\phi}{d\psi}(\psi) \right| d\xi d\psi \\ &= \frac{1}{4\pi} \left(\int_{\Psi(U)} \left| \frac{d\phi}{d\psi}(\psi) \right| d\psi \right) \left(\int_0^{2\pi} \alpha_i(g_{t+\tau} r_\xi X) d\xi \right) \\ &= \frac{1}{4\pi} \left(\int_U d\phi \right) (2\pi) \text{Ave}_{t+\tau}(\alpha_i)(X) \\ &\geq \frac{c'}{2} \text{Ave}_{t+\tau}(\alpha_i)(X) \end{aligned}$$

where we have used Lemma 5.1 for the last inequality. ■

5.6 Completing the proof

Proof of Proposition 5.3. The strategy is to use Proposition 5.4 repeatedly, moving out to a large circle, applying Lemma 5.2 for comparisons along the way. This will give an upper bound on the circle average that has one summand that is an exponentially decaying term times $\alpha_k(X)$, plus a mess of higher complexity terms and constants. At every stage, we accumulate higher complexity terms α_j , $j > k$. When we have done n stages, there are n higher complexity terms, each of which is a product of a higher complexity term from a circle corresponding to i steps, multiplied by $n - i$ decaying terms coming from Proposition 5.4. The i step circle part we control by downwards induction on k , and the $n - i$ decaying terms each decay a bit faster than a 1 step circle term would. The aggregate of the n higher complexity terms is thus bounded by a geometric series times $\sum_{j>k} \alpha_j(X)$, and the series decays around as fast as the $\alpha_k(X)$ term.

So we proceed by downwards induction on k . The statement is trivial when $k > M$ (recall that M is the complexity of the whole surface X), because for this k we have defined $\alpha_k(X) = 0$ for all X . So assume the result holds for all $j > k$, and we will prove it for k . For any $\tau > 0$, and non-negative integer n , we have, using Lemma 5.2, and then Proposition 5.4:

$$\begin{aligned} \text{Ave}_{n\tau}(\alpha_k)(X) &\leq c_2 \int_0^{2\pi} \text{Ave}_\tau(\alpha_k)(g_{(n-1)\tau} r_\theta X) d\theta \\ &\leq c_2 \int_0^{2\pi} \left(C e^{-\tau(1-\delta)} \alpha_k(g_{(n-1)\tau} r_\theta X) + w_\tau \sum_{j>k} \alpha_j(g_{(n-1)\tau} r_\theta X) + b_\tau \right) d\theta \\ &= 2\pi c_2 C e^{-\tau(1-\delta)} \text{Ave}_{(n-1)\tau}(\alpha_k)(X) + c_2 w_\tau \sum_{j>k} \int_0^{2\pi} \alpha_j(g_{(n-1)\tau} r_\theta X) d\theta + 2\pi c_2 b_\tau \end{aligned}$$

Applying the inductive hypothesis to the terms in the middle sum, and enlarging the constants b_τ, w_τ , we get

$$\text{Ave}_{n\tau}(\alpha_k)(X) \leq c \cdot e^{-\tau(1-\delta)} \text{Ave}_{(n-1)\tau}(\alpha_k)(X) + w_\tau \left(\sum_{j>k} (e^{-(1-2\delta)})^{(n-1)\tau} \alpha_j(X) \right) + b_\tau,$$

where the constant c is independent of τ . Now applying this inequality for $n-1, n-2, \dots, 1$ gives

$$\begin{aligned} \text{Ave}_{n\tau}(\alpha_k)(X) &\leq \left(c e^{-\tau(1-\delta)} \right)^n \alpha_k(X) \\ &\quad + w_\tau \sum_{j>k} \alpha_j(X) \left(e^{-\tau(1-2\delta)(n-1)} + e^{-\tau(1-2\delta)(n-2)} c e^{-\tau(1-\delta)} + e^{-\tau(1-2\delta)(n-3)} (c e^{-\tau(1-\delta)})^2 + \dots \right) \\ &\quad + b_\tau \left(1 + c e^{-\tau(1-\delta)} + (c e^{-\tau(1-\delta)})^2 + \dots \right). \end{aligned}$$

The common ratio in the first geometric series is $c e^{-\tau\delta}$, while in the second it is $c e^{-\tau(1-\delta)}$. By choosing τ sufficiently large, we can make both of these ratios less than $1/2$ (here we use that $\delta > 0$), and then each of the series is bounded by (twice) the first term. For new constants w_τ, b_τ , we get

$$\text{Ave}_{n\tau}(\alpha_k)(X) \leq \left(c e^{-\tau(1-\delta)} \right)^n \alpha_k(X) + w_\tau \cdot \sum_{j>k} \alpha_j(X) e^{-\tau(1-2\delta)(n-1)} + b_\tau.$$

We combine the terms, taking τ large enough that $c < e^{\tau\delta}$, (and choose new constants) to get

$$\text{Ave}_{n\tau}(\alpha_k)(X) \leq c_\tau e^{-\tau(1-2\delta)n} \sum_{j\geq k} \alpha_j(X) + b_\tau,$$

where c_τ is a constant depending on τ . Taking $T = n\tau$ gives the desired result. ■

6 Bounds on quadratic growth constants as function of genus

One can ask how the quadratic growth upper bound c_{\max} in Theorem 1.2 depends on the genus of the stratum. Our proof of the theorem can be modified to give c_{\max} that grows exponentially in the genus, and it seems hard to do better using our method. Some remarks on this are given below.

Giving a bound on the constant b in Proposition 5.3 will give an upper bound on c_{\max} (see proof of Theorem 1.2). We can get an upper bound on b that grows exponentially in the genus g as follows.

For the top complexity α_k , the best upper bound for b_t we can get in Proposition 5.4 is $f(t)\sqrt{g}^{1+\delta}$, where f is some function of t (the dependence on t doesn't really matter, since ultimately it will be fixed independent of genus). However in the proof of Proposition 5.3, when we move down in complexity, at every stage we get a constant term that is some constant times the constant terms for higher complexity. This ultimately gives exponential growth for the α_1 constant.

One might consider several additional improvements, which turn out to help slightly but still give an exponential bound. When one moves down in complexity, it is not necessary to add a constant times the sum of all the higher complexity terms, as we do in the proof. Instead, one should be able to add just the next highest complexity term. This involves two changes: (i) replacing the sum in Proposition 5.4 with a max, which can be done immediately, and (ii) replacing α_k with a function α'_k (essentially $\max_{i>k} \alpha_i$) that satisfies $\alpha'_k \geq \alpha'_{k+1}$. Change (ii) requires modifying the argument in several places, but none of the modifications are particularly difficult.

Note that if we work with quadratic differentials and allow simple poles (corresponding to points with cone angle π), then it is not clear that there is any bound for c_{\max} that depends only on the genus, since there are infinitely many strata in a given genus.

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